Virasoro constraints & Wall-crossing via vertex algebras

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1) Moduli of sheaves & pairs
2) Enumerative invariants
3) Wall-crossing
4) Virasoro constraints
5) Joyce's vertex algebras
6) Sketch of proof

<u>Thm</u> (Bojko – L – Moreira)

Virasoro constraints hold for $M_C(r,d)$, $M_S^{H\text{-}ss}(r,c_1,c_2)$.

$h^{0,1} = h^{0,2} = 0$.

① Moduli of sheaves & pairs

X smooth projective variety / $\mathbb{C}$

$$\mathrm{Coh}(X)_v \underset{\text{open}}{\supset} \mathrm{Coh}(X)_v^{H\text{-ss}} \xrightarrow[\text{good moduli}]{} M_X^{H\text{-ss}}(v)$$

F is <u>H-semistable</u> if $\forall\, G \subset F$, $\dfrac{P_G(x)}{r_G} \leq \dfrac{P_F(x)}{r_F}$

e.g. $M_C(r,d)$, $M_S^{H\text{-ss}}(r, c_1, c_2)$, $M_S^{H\text{-ss}}(0, \beta, \chi)$, $M_X^{H\text{-ss}}(v)$

Fano 3-fold.

Similarly, we have moduli of pairs $E \xrightarrow[\phi]{} F$

fixed

e.g. $C^{[n]}$, $P_C^{6\text{-ss}}(r,d)$, $\mathrm{Quot}_S(E, \beta, \chi)$, $\cancel{PT_X(\beta, \chi)}$.

② Enumerative invariants

$$M = M_X^{H\text{-ss}}(v) \overset{\bigstar}{\rightsquigarrow} \begin{array}{ccc} H^*(M,\mathbb{C}) & \otimes & H_*(M,\mathbb{C}) \longrightarrow \mathbb{C} \\ \uparrow \xi_{\mathbb{F}} & & \Psi \\ \mathbb{D}^X & & [M]^{\mathrm{vir}} \end{array}$$

Assumptions $\ast$

1) no strictly semistable objects in M

2) $\mathrm{Ext}^i(F,F)=0 \quad \forall i \geq 3, \ \forall F \in M$

3) $\exists$ universal sheaf $\mathbb{F}$ on $M \times X$

$\mathrm{vdim} = \mathrm{ext}^1 - \mathrm{ext}^2$

$= 1 - \chi(F,F)$

$$\xi_{\mathbb{F}} : \mathbb{D}^X \longrightarrow H^*(M, \mathbb{C})$$

$$\mathrm{ch}_i(\gamma) \longmapsto \pi_{1*}\left(\mathrm{ch}_{i+\dim X - p}\,\mathbb{F} \cup \pi_2^*\gamma\right)$$

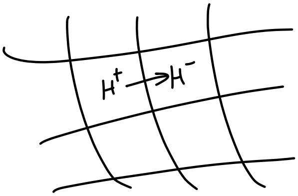
$i \geq 0, \ \gamma \in H^{p,*}(X)$

<u>Def</u> Descendent invariants are $\displaystyle\int_{[M]^{vir}} \xi_{\mathbb{F}}(D) \in \mathbb{C}$.

③ Wall-crossing

<u>Q</u>. How does moduli of sheaves / pairs change as we vary stability condition?

<u>Simple wall-crossing</u> :



1) <u>Geometry</u>

$$\begin{tikzcd}
 & & \tilde{M} \arrow[dl] \arrow[dr] & & \\
\mathbb{P}^+ \arrow[r, hook] \arrow[ddrr] & M^+ \arrow[dr] & & M^- \arrow[dl] & \mathbb{P}^- \arrow[l, hook'] \arrow[ddll] \\
 & & M^0 & & \\
 & & M(v_1) \times M(v_2) \arrow[u, hook] & &
\end{tikzcd}$$

2) Invariants : $\displaystyle\int_{M^+} \xi_{\mathbb{F}^+}(D) - \int_{M^-} \xi_{\mathbb{F}^-}(D) = \int_{M(v_1)\times M(v_2)}$

3) Joyce : $[M^+]^{vir} - [M^-]^{vir} = \Big[[M(v_1)]^{vir}, [M(v_2)]^{vir} \Big] \in$ where?

Key example (Thaddeus pairs)

$$\sigma \in \mathbb{R}_{>0} \rightsquigarrow P^\sigma := P_C^{\sigma\text{-ss}}(2,d) \quad \text{parametrizing}$$

(odd degree $d \gg 0$)

$$\mathcal{O}_C \xrightarrow{s} F \quad \text{s.t.} \quad \begin{cases} \operatorname{ch} F = (2,d) \\ \forall\, G \subseteq F, \quad \dfrac{d_G}{r_G} \le \dfrac{d+\sigma}{2} \\ \forall\, G \subsetneq F, \quad \dfrac{d_G+\sigma}{r_G} \le \dfrac{d+\sigma}{2} \\ \quad \mathcal{O}_C \subseteq G \end{cases}$$

$1 \quad 3 \quad \dots \quad d-2 \quad d \quad d+2 \quad \dots \qquad \sigma \in \mathbb{R}_{>0}$

$$P_0 \leftrightarrow P_1 \leftrightarrow \cdots \leftrightarrow P_n \leftrightarrow \underbrace{P_{n+1} = P_{n+2} = \cdots = P_\infty = \emptyset}_{\text{simple!}}$$

$$P_0 \xrightarrow{\pi} M_C(2,d)$$

$$* \quad [P_0] = \underbrace{[P_\infty]}_{\text{simple}} + \sum_{i\ge 0}^{\infty} \underbrace{[P_i] - [P_{i+1}]}_{\text{wall-crossing}}$$

later

$$* \quad \pi_*\big(c_{top}(T_\pi) \cap [P_0]\big) = \chi(F)\cdot [M(2,d)].$$

④ Virasoro constraints

Virasoro operators $\quad \mathbf{L_n} = R_n + T_n \curvearrowright \mathbb{D}^X, \quad n \geq -1$

* $\mathbf{R_n}$: derivation s.t.

$$R_n\left(ch_i^H(\gamma)\right) = i(i+1)\cdots(i+n)\, ch_{i+n}^H(\gamma)$$

* $\mathbf{T_n}$: multiplication by

$$T_n = \sum_{i+j=n} i!\,j! \sum_s (-1)^{\dim X - p_s^L}\, ch_i^H(\gamma_s^L)\, ch_j^H(\gamma_s^R)$$

$$\Delta_* \operatorname{td}(X) = \sum_s \gamma_s^L \otimes \gamma_s^R$$

Indeed, $[L_n, L_m] = (m-n) L_{n+m}$.

Naively ... $\displaystyle\int_{[M]^{vir}} \xi_{\mathbb{F}}\left(L_n(D)\right) = 0$ Wrong !

There are two ways to correct the formulation.

1) Correction operator [MOOP, M, vB]

Conjecture $\displaystyle\int_{[M]^{vir}} \xi_{\mathbb{F}}\left(\left(L_n + \tfrac{1}{r} S_n\right) D\right) = 0 \quad \forall n \geq -1,\ \forall D$

$$M = PT_X(\beta, \alpha),\ S^{[n]},\ M_S^{H\text{-}ss}(r, c_1, c_2).$$

2) Combine $\{L_n\}_{n\geq -1}$ to define

<u>Def</u> (BLM) $\mathsf{L}_{\mathrm{wt}_0} := \sum_{n=-1}^{\infty} \frac{(-1)^n}{(n+1)!} L_n \circ L_{-1}^{n+1} : \mathbb{D}^X \longrightarrow \mathbb{D}^X_{\mathrm{wt}_0}$

$$\mathbb{D}^X_{\mathrm{wt}_0} = \ker(L_{-1})$$

<u>Conjecture</u> (BLM) $\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{wt}_0}(D) = 0 \quad \forall D$.

↳ new for $M_C(r,d)$, $M_S^{H\text{-}ss}(0,\beta,\chi)$.

<u>Thm</u> (BLM) Two formulations are equivalent.

e.g. What does Virasoro conjecture really say?

$M = M_C(2,L) \rightsquigarrow \alpha, \beta, \gamma \quad \mathrm{degree}_C = 1, 2, 3$.

Virasoro $\Rightarrow (g-p)\int_M \alpha^m \beta^k \gamma^p = -2m \int_M \alpha^{m-1}\beta^{k-1}\gamma^{p+1}$.

where $m + 2k + 3p = 3g-3$.

※ follows directly from known formulas.

※ difficult to identify what it means for $M_C(3,d)$.

⑤ Joyce's vertex algebra

X smooth proj $\rightsquigarrow$ $\mathcal{M}_X$ higher stack for $D^b(X)$

<u>Thm</u> (Joyce) $V = H_*(\mathcal{M}_X, \mathbb{C})$ is naturally a vertex algebra.

$$\left(V, \mathbb{1}, T, Y(-,z) \right)$$

$0 \in \mathcal{M}_X$

$B\mathbb{G}_m \times \mathcal{M}_X \longrightarrow \mathcal{M}_X$

· $\Sigma : \mathcal{M}_X \times \mathcal{M}_X \longrightarrow \mathcal{M}_X$

· $R\mathcal{H}om_{\pi_{12}}(\mathcal{F}_1, \mathcal{F}_2)$ on $\mathcal{M}_X \times \mathcal{M}_X$.

$T : V \to V$, $Y(-,z) : V \longrightarrow \mathrm{End}(V)[\![z, z^{-1}]\!]$

$$a \longmapsto \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

Joyce's vertex algebra $V = H_*(\mathcal{M}_X, \mathbb{C})$

<u>Algebra</u>		<u>Geometry</u>
1) $\left(\check{V} = V/T(V), [\,,\,] \right)$	....	wall-crossing
2) $\omega \in V$	.....	Virasoro constraints.

1) $M = M_X^{H\text{-ss}}(v) \xhookrightarrow{i} \mathcal{M}_X^{\mathrm{ris}}$

$\Rightarrow \quad i_*[M]^{\mathrm{vir}} \in H_*(\mathcal{M}_X^{\mathrm{ris}}, \mathbb{C})$ " $=$ " $V/T(V)$

$\check{V} := V/T(V)$ has Lie algebra structure !?.

<u>Thm</u> (Joyce) Wall-crossing formulas via $(\check{V}, [\,,\,])$

when X = curve, surface.

2) $\omega \in V$ <u>conformal</u> if $\quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies

* L_0 diagonalizable
* $L_{-1} = T$
* $[L_n, L_m] = (n-m) L_{n+m} + \frac{(n^3-n)}{12} c \, \delta_{n+m,0}$.

<u>Def/Thm</u> (Borcherds) $P_i := \{ a \in V_i \mid L_n a = 0 \ \forall n \geq 1 \}$

$P_1 / P_0 \subset \check{V}$ Lie subalgebra of primary states.

Assume X : curve, surface w/ $h^{0,2}=0$, rational Fano 3-fold.

<u>Thm</u> (BLM) $M = M_X^{H\text{-ss}}(v)$ satisfies Virasoro conjecture

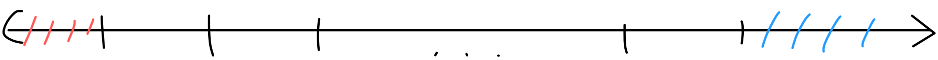
$$\iff i_*[M]^{vir} \in P_1/P_0 \subset \check{V} \text{ with respect to}$$

a natural conformal element $\omega \in V^{pa}$.

↳ Virasoro is compatible w/ wall-crossing.

↳ Sheaf-theoretic origin of Virasoro conjecture w/o reference to GW theory.

⑥ Sketch of proof (pair wall-crossing, pair Virasoro)



② $P_C^{0+}(r,d) \xleftrightarrow[t\in\mathbb{R}_{>0}]{}$ ① $P_C^{\infty}(r,d) = \begin{cases} \emptyset & r>0 \\ C^{[d]} & r=1 \end{cases}$

projective bundle compatibility

③ $\gamma_{(r,d)} \overset{\text{def}}{:=} f_*\left(c_{top}(T_f) \cap [P_C^{0+}(r,d)]^{vir} \right)$

$$= \underbrace{\chi(r,d)}_{>0} \underbrace{[M_C(r,d)]^{inv}}_{④} + (\text{lower rank})$$

Future : 1) moduli of objects in $Ku(X) \subset D^b(X)$

2) moduli of quiver rep'n $M_Q^{\theta-ss}(\underline{d})$

3) $PT_X(\beta, \chi)$ Fano 3-fold.

4) Why such phenomenoa ?